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Classification of spherical varieties

Paolo BRAVI

Abstract

We give a short introduction to the problem of classification of spherical varieties, by presenting the Luna conjecture about the classification of wonderful varieties and illustrating some of the related currently known results.

This is a revised version of notes of lectures given at the conference “Actions Hamiltoniennes : Invariants et Classification” (CIRM Luminy, April 2009) on the classification of spherical varieties.

Before these one could read the lecture notes of M. Brion [Bri] and of G. Pezzini [P] providing a first general introduction to the theory of spherical and wonderful varieties. Some of the most recent results about the classification, which are only mentioned here, are more extensively illustrated in the lecture notes of I.V. Losev [Lo].

Here we mainly focus on the Luna conjecture about the classification of wonderful varieties, which implies the classification of general spherical varieties. Our aim is just to present the conjecture and illustrate some of the related currently known results, without proofs.

Let me just say a few words about the state of the art in the classification of wonderful varieties.

The Luna conjecture is officially still open in general, that is, there is not yet any complete proof on the literature, but a definitive solution is being achieved by S. Cupit-Foutou ([C08, C09]), via invariant Hilbert schemes.

Another complete proof has been announced by G. Pezzini and the author ([BP09]), this proof relies on the original constructive approach of D. Luna.

We will not try to give a full description of such proofs, here we prefer to refer to the definitive versions of the above cited papers, which will likely appear soon.

In Section 1 we introduce the objects in use: wonderful varieties and their combinatorial counterpart, the spherical systems; we state the Luna conjecture and some of the related partial results.

In Section 2 we recall the definition of invariant Hilbert schemes and explain their interrelation with wonderful varieties.

In Section 3 we give some further insights into the relations between wonderful varieties (more precisely their generic isotropy groups) and spherical systems.

In Appendix A, for reader’s convenience, we place some remarks on two different definitions of the so-called spherical roots occurring in the literature.

In Appendix B we give the statement of the classification of general spherical varieties, assuming the Luna conjecture.

1. WONDERFUL VARIETIES

The basic reference for this section is [Lu01] (see also [P], Section 3).

1.1. Definition of wonderful varieties. Let G be a semisimple complex algebraic group, T a maximal torus, B a Borel subgroup containing T , S the corresponding set of simple roots of the root system of (G, T) .

Course taught during the meeting “Hamiltonian Actions: invariants et classification” organized by Michel Brion and Thomas Delzant. 6-10 April 2009, C.I.R.M. (Luminy).

Definition 1.1. A G -variety is called wonderful of rank r if it is smooth, complete, with r smooth prime G -divisors $X^{(1)}, \dots, X^{(r)}$ with normal crossings, such that, for any $I \subseteq \{1, \dots, r\}$, $\cap_{i \in I} X^{(i)}$ is a (non-empty) G -orbit closure.

Notice that by definition, for $I = \emptyset$, the whole wonderful G -variety X is requested to be a G -orbit closure, i.e. to contain an open (dense) G -orbit. For any $I \subseteq \{1, \dots, r\}$, the G -orbit closure $X_I = \cap_{i \in I} X^{(i)}$ of X is a wonderful G -variety by itself, with rank $X_I = r - \text{card } I$: the prime G -divisors of X_I are just $X_I^{(i)} = X^{(i)} \cap X_I$ for all $i \notin I$. In particular, the unique closed G -orbit, $\cap_{i=1}^r X^{(i)}$, is a wonderful variety of rank 0. By definition, a wonderful variety of rank 0, being homogeneous and complete, is a (generalized) flag variety, i.e. by Borel's Theorem it contains a (unique) point fixed by a Borel subgroup. Furthermore, recall that X is necessarily projective, since it is complete and contains a unique closed G -orbit.

In [P], Section 3, several examples of wonderful varieties are given. Here are two further examples.

Example 1.2. Let X be the projective variety

$$\{(P, \ell, P', \ell') : P, P' \text{ points } \ell, \ell' \text{ lines in } \mathbb{P}^2 \text{ s.t. } P \in \ell, P' \in \ell, P \in \ell'\}.$$

The group $G = \text{SL}(3)$ acts on \mathbb{P}^2 and clearly on X . With this action, X is a wonderful G -variety of rank 2.

Example 1.3. The variety X^n of complete quadrics in \mathbb{P}^n is a wonderful $\text{SL}(n+1)$ -variety of rank n . The case of $n = 2$ has already been defined in detail in [P], Example 3.4.5. Let us briefly recall the general set-theoretical definition ([DP, DGMP]). Notice that the non-degenerate quadrics (non-singular degree 2 hypersurfaces) in \mathbb{P}^n form a homogeneous space

for the action of $\text{SL}(n+1)$, this is the open $\text{SL}(n+1)$ -orbit of X^n . Furthermore, the singular locus $\text{sing}(Q)$ of a degenerate quadric Q in \mathbb{P}^n is a (proper) projective subspace of \mathbb{P}^n (here we adopt the convention that the singular locus of a double hyperplane in \mathbb{P}^n is the hyperplane itself): so, if $\dim \text{sing}(Q) = d > 0$, one can consider the set of quadrics (degree 2, dimension $d-1$) in $\text{sing}(Q)$. Now, a complete quadric in \mathbb{P}^n , i.e. a point x of the variety X^n , consists by definition in a "complete" series of quadrics everyone lying in the singular locus of the preceding one, that is, $Q_0, \dots, Q_{l(x)}$ where

- Q_0 is a quadric in \mathbb{P}^n , with $\dim \text{sing}(Q_0) > 0$ if $l(x) > 0$,
- for all i , $0 < i < l(x)$, Q_i is a quadric in $\text{sing}(Q_{i-1})$ with $\dim \text{sing}(Q_i) > 0$,
- $Q_{l(x)}$ is a quadric in $\text{sing}(Q_{l(x)-1})$ that is non-degenerate or such that $\dim \text{sing}(Q_{l(x)}) = 0$.

Theorem 1.4 ([Lu96]). *Any wonderful variety is spherical.*

If a spherical homogeneous space, say G/H , admits an embedding that is a wonderful G -variety, then this embedding is unique (up to G -equivariant isomorphism) and corresponds to the canonical embedding. In particular, the cone of G -invariant valuations is strictly convex and $N(H)/H$ is finite.

1.2. The spherical system of a wonderful variety. Denote by $\Lambda(X)$ the lattice of weights of B -semiinvariant rational functions on X .

Let X be a wonderful G -variety, embedding of G/H . Denote by z the (unique) point in X stabilized by B_- (where B_- is the Borel subgroup opposite to B with respect to T , i.e. $B_- \cap B = T$). Clearly z lies in the closed G -orbit.

For us here the spherical roots of X are by definition equal to the T -weights occurring in the normal space $T_z X / T_z G.z$, also equal (up to a minus sign) to the equations, primitive in $\Lambda(X)$, of the (hyper-)faces of the cone of G -invariant valuations. They are linearly independent and form a basis of $\Lambda(X)$.

Denote by Σ_X the set of spherical roots of X . Clearly, they are in correspondence with prime G -divisors.

We will comment this definition of spherical roots later in Appendix A.

Denote by Δ_X the set of colors of X , the not G -stable prime B -divisors of X . They correspond to the closures in X of the colors of G/H .

It is possible to define a \mathbb{Z} -bilinear pairing, called Cartan pairing,

$$c_X: \mathbb{Z}\Delta_X \times \Lambda(X) \rightarrow \mathbb{Z}$$

such that, for $D \in \Delta_X$, $c_X(D, -)$ corresponds to the discrete valuation associated to the divisor D on $\mathbb{C}(X)$.

For all $\alpha \in S$, set

$$\Delta_X(\alpha) = \{D \in \Delta_X : P_{\{\alpha\}} \cdot D \neq D\},$$

where $P_{\{\alpha\}}$ is the minimal parabolic subgroup containing B associated to α . One has $\cup_{\alpha \in S} \Delta_X(\alpha) = \Delta_X$ and recall that, for all $\alpha \in S$, $\text{card}(\Delta_X(\alpha)) \leq 2$.

Set $S_X^p = \{\alpha \in S : \Delta_X(\alpha) = \emptyset\}$, then the parabolic subgroup P_X containing $P_{\{\alpha\}}$, for all $\alpha \in S_X^p$, is the stabilizer of the open B -orbit, equal to the parabolic subgroup opposite to G_z with respect to T .

If $D \in \Delta_X(\alpha)$ and $\text{card}(\Delta_X(\alpha)) = 1$ then $c_X(D, -)$ is uniquely determined by α : if $2\alpha \in \Sigma_X$ then $c_X(D, -) = \frac{1}{2}\langle \alpha^\vee, - \rangle$; otherwise $c_X(D, -) = \langle \alpha^\vee, - \rangle$.

Last, $\text{card}(\Delta_X(\alpha)) = 2$ if and only if $\alpha \in \Sigma_X$, in this case, say $\Delta_X(\alpha) = \{D^+, D^-\}$, $c_X(D^+, -) + c_X(D^-, -) = \langle \alpha^\vee, - \rangle$. Denote by A_X an abstract set, in correspondence with the subset of colors $D \in \cup_{\alpha \in S \cap \Sigma_X} \Delta_X(\alpha)$, endowed with the elements $c_X(D, -)$ of $\Lambda(X)^*$.

Definition 1.5. The datum of $\mathcal{S}_X = (S_X^p, \Sigma_X, A_X)$ is called the spherical system of X .

The spherical system of X carries some information on the variety X : the dimension of X is given by $\text{rank } X + \dim G.z$ (i.e. $\text{card } \Sigma_X$ plus the number of positive roots of G not belonging to the root subsystem generated by S_X^p).

Some properties of the generic stabilizer H of X , such as being reductive, very reductive in G (i.e. contained in no proper parabolic subgroup) or very solvable in G (i.e. contained in a Borel subgroup), can also be read off the spherical system \mathcal{S}_X . In particular, the rank of the character group of H , namely the dimension of its connected center, is equal to $\text{card } \Delta_X - \text{card } \Sigma_X$.

Example 1.6. Let X be the wonderful variety of Example 1.2. Then $S_X^p = \emptyset$, $\Sigma_X = S = \{\alpha_1, \alpha_2\}$, $A_X = \{D_1^+, D_1^-, D_2^+, D_2^-\}$ and the Cartan pairing is as follows.

	α_1	α_2
D_1^+	1	0
D_1^-	1	-1
D_2^+	0	1
D_2^-	-1	1

Example 1.7. Let X^n be the wonderful variety of Example 1.3. Then $S_X^p = \emptyset$, $\Sigma_X = 2S = \{2\alpha_1, \dots, 2\alpha_n\}$, $\Delta_X = \{D_1, \dots, D_n\}$ and the Cartan pairing is $c(D_i, 2\alpha_j) = \langle \alpha_i^\vee, \alpha_j \rangle$.

1.3. Classification of wonderful varieties. The set of spherical roots $\Sigma' = \Sigma_{X'}$ of a G -orbit closure X' of X is a subset of Σ_X , and X' is called the localization of X with respect to Σ' and denoted by $X' = X_{\Sigma'}$. The spherical system of X' is given by $S_{X'}^p = S_X^p$, $\Sigma_{X'} = \Sigma'$ and $A_{X'}$, and one can prove that $A_{X'}$ can be identified with the subset of colors $D \in \cup_{\alpha \in S \cap \Sigma'} \Delta_X(\alpha)$ and set $c_{X'}(D, \sigma) = c_X(D, \sigma)$ for all $\sigma \in \Sigma'$.

In particular, any spherical root of X is the spherical root of a wonderful G -variety of rank 1. The wonderful G -varieties of rank 1 are finitely many (up to G -equivariant isomorphism) and well known, for all G (see [A]). The finite set of spherical roots of wonderful G -varieties of rank 1 is denoted by $\Sigma(G)$.

The spherical system of X is actually determined by the spherical systems of all the localizations $X_{\Sigma'}$ of rank 2 (actually, it is enough to restrict to the localizations of rank 1 and those of rank 2 with a simple spherical root). Furthermore, the wonderful G -varieties of rank 2 are finitely many (up to G -equivariant isomorphism) and known, for all G (see [W]).

In the above cited papers the spherical systems of rank 1 and rank 2 wonderful varieties are essentially given, and it follows that two wonderful G -varieties of rank $r \leq 2$ with the same spherical system are G -equivariantly isomorphic.

Here we give a first quick, quite involved, definition of abstract spherical systems, in Section 3 we will give the original axiomatic definition due to D. Luna.

Definition 1.8.

- A spherical G -system of rank $r \leq 2$ is the spherical system of a wonderful G -variety of rank r .
- A spherical G -system of rank $r > 2$ is a triple $\mathcal{S} = (S^p, \Sigma, A)$, where $S^p \subset S$, $\Sigma \subset \Sigma(G)$ with $\text{card } \Sigma = r$ and a set $A = \cup_{\alpha \in S \cap \Sigma} A(\alpha)$, endowed with a \mathbb{Z} -bilinear pairing $c: \mathbb{Z}A \times \mathbb{Z}\Sigma \rightarrow \mathbb{Z}$, such that for all $\Sigma' \subset \Sigma$ with $\text{card } \Sigma' = 2$ the triple S^p , Σ' and $A_{\Sigma'} = \cup_{\alpha \in S \cap \Sigma'} A(\alpha)$, with the restriction of c , is a spherical G -system of rank 2.

Conjecture 1.9 ([Lu01]). *Wonderful G -varieties are classified by spherical G -systems.*

1.4. References of related known results.

Affine spherical homogeneous G -spaces (or equivalently reductive spherical subgroups of G) are known, for all G , [Kr, Mi, Bri87]. Those which admit a wonderful completion can be explicitly listed and the corresponding spherical systems are known (see [BP09]).

Two wonderful G -varieties with the same spherical system are G -equivariantly isomorphic, this has been proved by I.V. Losev in [Lo09]. Actually, Losev's Uniqueness Theorem is more generally stated in terms of spherical homogeneous \tilde{G} -spaces. Two smooth affine spherical \tilde{G} -varieties with the same weight monoid are \tilde{G} -equivariantly isomorphic, [Lo10]. Both results are presented more in detail in [Lo].

Let G be of adjoint type. The above conjecture has been proved in case the Dynkin diagram of the root system of G is simply laced [Lu01, BP05, Bra]. Regardless of the type of the root system, it has also been proved that strict wonderful G -varieties are classified by a certain subclass of spherical G -systems [BC10]: a wonderful G -variety is called strict if all its isotropy groups are selfnormalizing; it is strict if and only if can be realized as G -subvariety in a simple projective G -space [P07]. Recently, the same approach to the proof of the classification has been adapted to the general case, [BP09].

Let \tilde{G} be a connected reductive algebraic group. If the conjecture holds for the adjoint group $G = \tilde{G}/C_{\tilde{G}}$, then spherical homogeneous \tilde{G} -spaces (and then also spherical \tilde{G} -varieties) are classified by combinatorial objects ([Lu01]), this will be better stated in Appendix B. We can actually restrict to the classification of spherically closed subgroups (see 3.4 for the definition of spherical closure).

Recently, a full intrinsic proof of the above conjecture has been proposed by S. Cupit-Foutou, [BC08, C08, C09], via invariant Hilbert schemes, see Section 2.

2. INVARIANT HILBERT SCHEMES

2.1. Definition of invariant Hilbert schemes. Here we recall freely results from [AB]. Invariant Hilbert schemes can be defined for all not necessarily connected reductive groups G and all rational G -modules R with finite multiplicities: here we will restrict to the spherical case and give definitions just in this case, as follows.

Let G be connected (and reductive), T a maximal torus, B a Borel subgroup containing T and U the unipotent radical of B (a maximal unipotent subgroup of G). Let V be a finite dimensional G -module and Γ a (finitely generated) submonoid of Λ^+ fulfilling the extra condition, $\mathbb{Q}_{>0}\Gamma \cap \mathbb{Z}\Gamma = \Gamma$, corresponding to the normality property (see [Bri], Theorem 2.14).

Let us give a preliminary simplified definition. Let \mathbb{A}^1 be the affine line, $\mathbb{C}[\mathbb{A}^1] = \mathbb{C}[t]$.

A closed G -subvariety $\mathcal{X} \subset V \times \mathbb{A}^1$, such that the projection $\pi: \mathcal{X} \rightarrow \mathbb{A}^1$ is G -invariant, is called family of closed G -subvarieties of V over \mathbb{A}^1 .

It is called of type Γ if, decomposing the $\mathbb{C}[t]$ - G -algebra $R = \mathbb{C}[\mathcal{X}]$ as $\mathbb{C}[t]$ - G -module one has

$$R \cong \bigoplus_{\lambda \in \Gamma} R_{\lambda}^U \otimes V(\lambda),$$

and R_{λ}^U is a free $\mathbb{C}[t]$ -module of rank 1, for all $\lambda \in \Gamma$.

In particular, π is flat, surjective and maps closed G -subsets to closed subsets. Indeed, $\pi//G: \mathcal{X}//G \rightarrow \mathbb{A}^1$ is an isomorphism.

More easily, fix a point t in \mathbb{A}^1 , set $X_t = \pi^{-1}(t) \subset V$, G -stable subvariety, then X_t is spherical with weight monoid $\Lambda^+(X_t) = \Gamma$, for all t .

Example 2.1. Consider the vector space V of quadratic forms in n variables, with the action of $G = \text{SL}(n)$ by change of variables. Set $\mathcal{X} = V$, $\pi: V \rightarrow \mathbb{A}^1$ given by the discriminant. Then

\mathcal{X} is a family of closed G -subvarieties of V of type $\Gamma = \mathbb{N}\langle 2\omega_1, \dots, 2\omega_{n-1} \rangle$ over \mathbb{A}^1 (see [Bri], Examples 1.27.3 and 2.11.4).

Definition 2.2. Let Z be a scheme, a closed G -subscheme $\mathcal{X} \subset V \times Z$, such that the projection $\pi: \mathcal{X} \rightarrow Z$ is G -invariant, is called family of closed G -subschemes of V over Z . It is called of type Γ if, decomposing the sheaf of \mathcal{O}_Z - G -algebras $\mathcal{R} = \pi_* \mathcal{O}_{\mathcal{X}}$ as sheaf of \mathcal{O}_Z - G -modules one has

$$\mathcal{R} \cong \bigoplus_{\lambda \in \Gamma} \mathcal{R}_{\lambda}^U \otimes V(\lambda),$$

and \mathcal{R}_{λ}^U is an invertible sheaf of \mathcal{O}_Z -modules, for all $\lambda \in \Gamma$.

In particular, π is flat, surjective and maps closed G -subsets to closed subsets. Indeed, $\pi//G: \mathcal{X}//G \rightarrow Z$ is an isomorphism.

There is a natural contravariant functor assigning to any scheme Z the set of families \mathcal{X} of closed G -subschemes of V of type Γ over Z : given a morphism $Z_1 \rightarrow Z_2$ and a family \mathcal{X}_2 of closed G -subschemes of V of type Γ over Z_2 , the pull-back of \mathcal{X}_2 is a family of closed G -subschemes of V of type Γ over Z_1 .

This functor is representable, that is: there exists a scheme $\text{Hilb}_{\Gamma}^G(V)$, called invariant Hilbert scheme, and a family $\text{Univ}_{\Gamma}^G(V)$ of closed G -subschemes of V over $\text{Hilb}_{\Gamma}^G(V)$, called universal family, such that any family of closed G -subschemes of V of type Γ over Z is the pull-back of $\text{Univ}_{\Gamma}^G(V)$ through a morphism $Z \rightarrow \text{Hilb}_{\Gamma}^G(V)$.

Theorem 2.3 ([HS, AB]). *The scheme $\text{Hilb}_{\Gamma}^G(V)$ is quasiprojective.*

A spherical G -subvariety X of V with weight monoid $\Lambda^+(X) = \Gamma$ can thus be regarded as a closed point of $\text{Hilb}_{\Gamma}^G(V)$.

In general, let X_0 and X_1 be two affine G -varieties, X_0 is said to be a G -equivariant degeneration of X_1 , or X_1 to be a G -equivariant deformation of X_0 , if $\mathbb{C}[X_0]$ is G -isomorphic (as algebra) to the graded algebra associated to a filtration of $\mathbb{C}[X_1]$.

The G -subvariety X of V is called non-degenerate if its projections to the isotypical components of V are non-trivial.

Choose pairwise distinct generators $\lambda_1, \dots, \lambda_s$ of Γ , and set

$$V = V(\lambda_1)^* \oplus \dots \oplus V(\lambda_s)^*.$$

In this case non-degenerate spherical G -subvarieties of V with weight monoid Γ are the closed points of an open subscheme M_{Γ} of $\text{Hilb}_{\Gamma}^G(V)$.

Theorem 2.4. *The scheme M_{Γ} does not depend on the choice of the set of generators of Γ . It is affine, of finite type.*

Let $v_{\lambda_i^*}$ be a highest weight vector in $V(\lambda_i)^*$, for all $i = 1, \dots, s$. Define X_0 to be the closure of $G.v_0$ for $v_0 = v_{\lambda_1^*} + \dots + v_{\lambda_s^*}$, this is a horospherical variety with weight monoid $\Lambda^+(X_0) = \Gamma$, non-degenerate in V : a closed point in M_{Γ} . Each closed point of M_{Γ} corresponds to a G -equivariant deformation of X_0 .

The scheme M_{Γ} can be directly defined as follows. Let Y be an affine T -variety of weight monoid Γ . A family of affine G -schemes of type Y over Z is a family of affine G -schemes $\pi: \mathcal{X} \rightarrow Z$ together with an isomorphism $\mathcal{X}//U \rightarrow Y \times Z$ of families of affine T -schemes over Z .

Theorem 2.5. *The contravariant functor assigning to a scheme Z the set of equivalence classes of families of affine G -schemes of type Y over Z is represented by M_{Γ} .*

The affine scheme M_{Γ} parametrizes G -equivariant multiplication laws on the G -module

$$R = \bigoplus_{\lambda \in \Gamma} R_{\lambda}^U \otimes V(\lambda)$$

extending the given T -equivariant multiplication law in $R^U = \mathbb{C}[Y]$.

A G -equivariant multiplication law on R can be seen as a G -equivariant morphism $m: R \otimes R \rightarrow R$, satisfying some associativity and commutativity conditions.

A G -equivariant morphism $m: R \otimes R \rightarrow R$ is the sum of

$$m_\nu^{\lambda,\mu}: V(\lambda) \otimes V(\mu) \rightarrow V(\nu),$$

for $\lambda, \mu, \nu \in \Gamma$ and $\lambda + \mu - \nu \in \mathbb{N}S$.

The natural action of the adjoint torus T_{ad} (the maximal torus T modulo the center of G) on M_Γ

$$t.m_\nu^{\lambda,\mu} = t^{\lambda+\mu-\nu} m_\nu^{\lambda,\mu}$$

admits a unique extension to a morphism $\mathbb{A}^S \times M_\Gamma \rightarrow M_\Gamma$.

Let us freely identify non-degenerate spherical G -subvariety of V with weight monoid Γ with closed points in M_Γ , as above. Two such subvarieties X_1, X_2 are G -isomorphic if and only if $T_{\text{ad}}.X_1 = T_{\text{ad}}.X_2$; X_1 is degeneration of X_2 if and only if $X_1 \in \overline{T_{\text{ad}}.X_2}$. The point X_0 , corresponding to the above defined horospherical G -subvariety, is T_{ad} -stable and forms the unique closed T_{ad} -orbit.

Theorem 2.6 ([Kn], Theorem 1.3). *Let X be an affine spherical G -variety with weight monoid Γ , let m be the G -equivariant morphism $m: \mathbb{C}[X] \otimes \mathbb{C}[X] \rightarrow \mathbb{C}[X]$ corresponding to the multiplication law. The saturation of the submonoid $\langle \lambda + \mu - \nu : m_\nu^{\lambda,\mu} \neq 0 \rangle \subset \mathbb{N}S$ is free.*

Its basis, say Σ , equals the set of spherical roots of X , up to some integer multiples (see Appendix A).

Therefore, the normalization of $\overline{T_{\text{ad}}.X}$ is isomorphic to \mathbb{A}^Σ .

The horospherical variety X_0 corresponds to the case where $m_\nu^{\lambda,\mu} = 0$ if $\nu \neq \lambda + \mu$.

Theorem 2.7. *There are only finitely many T_{ad} -orbits in M_Γ .*

2.2. The strict case. Assume now that Γ is Λ^+ -saturated (i.e. $\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma$) and free, say $\Gamma = \mathbb{N}\langle \lambda_1, \dots, \lambda_s \rangle$ with linearly independent λ_i 's.

As above, let $v_{\lambda_i^*} \in V(\lambda_i)^*$ be a highest weight vector, $i = 1, \dots, s$, and set $v_0 = v_{\lambda_1^*} + \dots + v_{\lambda_s^*}$, $X_0 = \overline{G.v_0}$ is the horospherical G -variety corresponding to the T_{ad} -stable point in $M_\Gamma \subset \text{Hilb}_\Gamma^G(V)$.

It is just the affine multicone over the flag variety

$$G/G_{x_0} \subset \mathbb{P}(V(\lambda_1)^*) \times \dots \times \mathbb{P}(V(\lambda_s)^*)$$

where

$$x_0 = ([v_{\lambda_1^*}], \dots, [v_{\lambda_s^*}]).$$

Set S^p to be the set of simple roots orthogonal to $\lambda_1, \dots, \lambda_s$.

Now, it is easy to see that $\text{codim}_{X_0}(X_0 \setminus G.v_0) \geq 2$ and one can prove that this implies

$$T_{X_0} \text{Hilb}_\Gamma^G(V) \cong (V/\mathfrak{g}.v_0)^{G_{v_0}}$$

The “normalized” T_{ad} -action on V ,

$$t.v = t^{\lambda-\mu} v, \quad v \in V(\lambda)_\mu^{(T)},$$

passes to $T_{X_0} \text{Hilb}_\Gamma^G(V)$ and corresponds to the differential of the T_{ad} -action defined above on M_Γ .

One can prove that $T_{X_0} \text{Hilb}_\Gamma^G(V)$ is a multiplicity-free T_{ad} -space. Set Σ to be the set of T_{ad} -weights in $T_{X_0} \text{Hilb}_\Gamma^G(V)$. The triple $\mathcal{S} = (S^p, \Sigma, A = \emptyset)$ is a “strict” spherical G -system.

Example 2.8. Compute this for $G = \text{SL}(3)$, $\lambda_1 = 2\omega_1$, $\lambda_2 = 2\omega_2$.

Theorem 2.9 ([BC08, BC10]).

- (1) *There exists a strict wonderful G -variety X with $\mathcal{S}_X = \mathcal{S}$,*

$$G/G_{x_0} \subset X \subset \mathbb{P}(V(\lambda_1)^*) \times \dots \times \mathbb{P}(V(\lambda_s)^*)$$

- (2) $M_\Gamma \cong \mathbb{A}^\Sigma$

- (3) $j_{M_\Gamma}^* \text{Univ}_\Gamma^G(V) \subset V \times \mathbb{A}^\Sigma$ equals the normalization of the affine multicone \tilde{X} over X , $X_0 \subset \tilde{X} \subset V$. One has $j_{M_\Gamma}^* \text{Univ}_\Gamma^G(V)/T_{\text{ad}} \cong X$.

By the way, X is therefore uniquely determined by \mathcal{S} , up to G -equivariant isomorphism.

Example 2.10. Let G and $\Gamma = \mathbb{N}\langle\lambda_1, \lambda_2\rangle$ be as in Example 2.8. Find a vector $v_1 \in V$ such that $X_1 = \overline{G.v_1}$ is a non-degenerate spherical G -subvariety with $\Lambda^+(X_1) = \Gamma$ and $\overline{T_{\text{ad}}.X_1} = M_\Gamma$.

3. SPHERICAL SYSTEMS AND WONDERFUL SUBGROUPS

3.1. Axiomatic definition of spherical systems. Let G be (semisimple) of adjoint type.

Definition 3.1. A spherical G -system is a triple $\mathcal{S} = (S^p, \Sigma, A)$ with

- $S^p \subset S$,
- Σ is a set of linearly independent elements of $\Sigma(G)$,
- A a finite set endowed with a \mathbb{Z} -bilinear pairing (the restricted Cartan pairing) $c: \mathbb{Z}A \times \mathbb{Z}\Sigma \rightarrow \mathbb{Z}$, for all $\alpha \in \Sigma \cap S$, set $A(\alpha) = \{D \in A : c(D, \alpha) = 1\}$,

such that:

- (A1) for all $D \in A$ and $\sigma \in \Sigma$, $c(D, \sigma) \leq 1$ and if $c(D, \sigma) = 1$ then $\sigma \in S$;
- (A2) for all $\alpha \in S \cap \Sigma$, $\text{card}(A(\alpha)) = 2$ and, if $A(\alpha) = \{D_\alpha^+, D_\alpha^-\}$, $c(D_\alpha^+, \sigma) + c(D_\alpha^-, \sigma) = \langle \alpha^\vee, \sigma \rangle$ for all σ ;
- (A3) $A = \cup_{\alpha \in S \cap \Sigma} A(\alpha)$;
- (Σ 1) if $2\alpha \in 2S \cap \Sigma$ then $\frac{1}{2}\langle \alpha^\vee, \sigma \rangle \in \mathbb{Z}_{\leq 0}$ for all $\sigma \in \Sigma \setminus \{2\alpha\}$;
- (Σ 2) if $\alpha, \beta \in S$, with $\alpha \perp \beta$ and $\alpha + \beta \in \Sigma$, then $\langle \alpha^\vee, \sigma \rangle = \langle \beta^\vee, \sigma \rangle$ for all σ ;
- (S) for all $\sigma \in \Sigma$, there exists a wonderful G -variety X of rank 1 with $\Sigma_X = \{\sigma\}$ and $S^p = S_X^p$.

The set $\Sigma(G)$ for any G (of adjoint type) is the set of $\sigma \in \mathbb{N}S$ such that

- $\sigma = \alpha + \beta$ for orthogonal $\alpha, \beta \in S$,
- or $\text{supp } \sigma$ generates an irreducible root subsystem and, after restricting S to $\text{supp } \sigma$, σ is one of the following:

type of supp	σ
$A_n, n \geq 1$	$\sum_{i=1}^n \alpha_i$
A_1	$2\alpha_1$
$B_n, n \geq 2$	$\sum_{i=1}^n \alpha_i$
$B_n, n \geq 2$	$\sum_{i=1}^n 2\alpha_i$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
$C_n, n \geq 3$	$\alpha_1 + (\sum_{i=2}^{n-1} 2\alpha_i) + \alpha_n$
$D_n, n \geq 3$	$(\sum_{i=1}^{n-2} 2\alpha_i) + \alpha_{n-1} + \alpha_n$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$2\alpha_1 + \alpha_2$
G_2	$4\alpha_1 + 2\alpha_2$
G_2	$\alpha_1 + \alpha_2$

The axiom (S) is equivalent to:

- $\{\alpha \in \text{supp } \sigma : \alpha \perp \sigma \text{ and } \sigma - \alpha \text{ is not a root}\} \subset S^p \subset \{\alpha \in S : \alpha \perp \sigma\}$, if $\text{supp } \sigma$ is of type B_n or C_n ,
- $\{\alpha \in \text{supp } \sigma : \alpha \perp \sigma\} \subset S^p \subset \{\alpha \in S : \alpha \perp \sigma\}$, if $\text{supp } \sigma$ is of type F_4 ,

or one of the two conditions (which are equivalent) in the other cases.

The set of colors Δ of \mathcal{S} is obtained extending A and c with colors

- $D_{2\alpha}$, for all $2\alpha \in 2S \cap \Sigma$, setting $c(D_{2\alpha}, \sigma) = \frac{1}{2}\langle \alpha^\vee, \sigma \rangle$ for all σ ,
- D_α , for all $\alpha \in S \setminus (S^p \cup \Sigma \cup \frac{1}{2}\Sigma)$, up to identifications $D_\alpha = D_\beta$ if $\alpha \perp \beta$ and $\alpha + \beta \in \Sigma$, setting $c(D_\alpha, \sigma) = \langle \alpha^\vee, \sigma \rangle$ for all σ .

Moreover, set $\Delta(\alpha) = A(\alpha)$ if $\alpha \in S \cap \Sigma$, $\Delta(\alpha) = \{D_{2\alpha}\}$ if $2\alpha \in 2S \cap \Sigma$, $\Delta(\alpha) = \{D_\alpha\}$ if $\alpha \in S \setminus (S^p \cup \Sigma \cup \frac{1}{2}\Sigma)$.

For any G there exist only finitely many spherical G -systems (recall that $\Sigma \subset \Sigma(G)$ and the latter is finite). They can be enumerated. For example, here we give the number of spherical G -systems, for all G simple of rank ≤ 4 .

Table 3.1

diagram	spherical root
	$\alpha_1 + \alpha'_1$
	α_1
	$\sum_{i=1}^n \alpha_i, n \geq 2$
	$2\alpha_1$
	$\sum_{i=1}^n \alpha_i, n \geq 2$
	$\sum_{i=1}^n 2\alpha_i, n \geq 2$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3$
	$\alpha_1 + (\sum_{i=2}^{n-1} 2\alpha_i) + \alpha_n, n \geq 3$
	$(\sum_{i=1}^{n-2} 2\alpha_i) + \alpha_{n-1} + \alpha_n, n \geq 3$
	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
	$2\alpha_1 + \alpha_2$
	$4\alpha_1 + 2\alpha_2$
	$\alpha_1 + \alpha_2$

	rk 0	rk 1	rk 2	rk 3	rk 4	tot
A ₁	2 ¹	2				4
A ₂	2 ²	5	3			12
B ₂	2 ²	7	8			19
G ₂	2 ²	7	5			16
A ₃	2 ³	15	17	10		50
B ₃	2 ³	19	25	24		76
C ₃	2 ³	17	23	18		66
A ₄	2 ⁴	37	50	50	40	193
B ₄	2 ⁴	45	76	89	83	309
C ₄	2 ⁴	42	71	80	58	267
D ₄	2 ⁴	41	63	75	32	227
F ₄	2 ⁴	41	61	77	71	266

3.2. Luna diagrams. One can represent spherical systems just via diagrams, by adding some lines and circles on the Dynkin diagram of the root system of (G, T) .

- Each spherical root has its own symbol to be drawn on the corresponding support, as in Table 3.1.
- A circle must be around the vertex of a simple root not in S^p , $S \cap \Sigma$ or $S \cap \frac{1}{2}\Sigma$.
- Each circle corresponds to a color, circles corresponding to the same color are joined by a line. For the color $D_{\alpha_i}^+$ corresponding to a circle over the vertex $\alpha_i \in S \cap \Sigma$ we can always suppose $c(D_{\alpha_i}^+, \sigma) \geq -1$ for all $\sigma \in \Sigma$, if $c(D_{\alpha_i}^+, \sigma) = -1$ for some $\sigma \in \Sigma$ with $\alpha_i \not\perp \sigma$, then there is an arrow, “<” or “>”, pointing towards $\text{supp } \sigma$.

Example 3.2.

- (1) For the spherical system of Example 1.6:



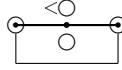
- (2) For the spherical system of Example 1.7:



- (3) G of type A_3 , $S^p = \emptyset$, $\Sigma = \{\alpha_1 + \alpha_3, \alpha_2\}$, $A = \{D_{\alpha_2}^+, D_{\alpha_2}^-\}$ with restricted Cartan pairing as follows:

	$\alpha_1 + \alpha_3$	α_2
$D_{\alpha_2}^+$	-1	1
$D_{\alpha_2}^-$	-1	1

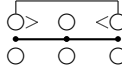
Then the digram is:



- (4) G of type A_3 , $S^p = \emptyset$, $\Sigma = S$, $A = \{D_{\alpha_1}^+, D_{\alpha_1}^-, D_{\alpha_2}^+, D_{\alpha_2}^-, D_{\alpha_3}^-\}$ with Cartan pairing as follows:

	α_1	α_2	α_3
$D_{\alpha_1}^+$	1	-1	1
$D_{\alpha_1}^-$	1	0	-1
$D_{\alpha_2}^+$	0	1	0
$D_{\alpha_2}^-$	-1	1	-1
$D_{\alpha_3}^-$	-1	0	1

Then the diagram is:



3.3. Wonderful morphisms.

Definition 3.3.

- Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical G -system with set of colors Δ . A subset of colors $\Delta^* \subset \Delta$ is called distinguished if there exists $D \in \mathbb{N}_{>0}\Delta^*$ such that $c(D, \sigma) \geq 0$ for all $\sigma \in \Sigma$.

If the monoid

$$\{\sigma \in \mathbb{N}\Sigma : c(D, \sigma) = 0, \forall D \in \Delta^*\}$$

is free, then the quotient of \mathcal{S} by Δ^* is the spherical G -system $\mathcal{S} = ({}^{\setminus}S^p, {}^{\setminus}\Sigma, {}^{\setminus}A)$, also denoted by \mathcal{S}/Δ^* , defined as:

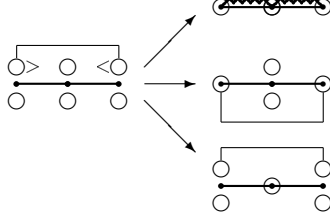
- ${}^{\setminus}S^p = \{\alpha \in S : \Delta(\alpha) \subset \Delta^*\}$,
- ${}^{\setminus}\Sigma$ basis of $\{\sigma \in \mathbb{N}\Sigma : c(D, \sigma) = 0, \forall D \in \Delta^*\}$,
- ${}^{\setminus}A = \cup A(\alpha)$ for all $\alpha \in S$ such that $A(\alpha) \cap \Delta^* = \emptyset$, with the same Cartan pairing restricted to $\mathbb{Z}{}^{\setminus}A \times \mathbb{Z}{}^{\setminus}\Sigma$.
- A G -equivariant surjective morphism $\phi: X \rightarrow \setminus X$ with connected fibers between wonderful G -varieties is called a wonderful G -morphism. The subset of colors of Δ_X that map dominantly onto $\setminus X$ is denoted by Δ_ϕ .

Proposition 3.4 ([Lu01], Proposition 3.3.2).

- Let $\phi: X \rightarrow \setminus X$ be a wonderful G -morphism, then Δ_ϕ is a distinguished subset of Δ_X and $\mathcal{S}_X = \mathcal{S}_X/\Delta_\phi$.

- Let X be a wonderful G -variety, if Δ^* is a distinguished subset of Δ_X with well defined quotient \mathcal{S}_X/Δ^* , then there exists a (unique) wonderful G -morphism $\phi: X \rightarrow \mathcal{S}_X/\Delta^*$ such that $\Delta_\phi = \Delta^*$.

Example 3.5. In Example 3.2 (4) above there are three minimal distinguished subsets of colors: $\{D_{\alpha_1}^+, D_{\alpha_2}^-\}$, $\{D_{\alpha_1}^-, D_{\alpha_3}^-\}$ and $\{D_{\alpha_2}^+\}$. The diagrams of the corresponding quotients appear in the following picture.



Let $\phi: X \rightarrow \mathcal{S}_X/\Delta^*$ be a wonderful G -morphism, if \mathcal{H} is a generic isotropy group of \mathcal{S}_X/Δ^* then a generic isotropy group of X , say H , can be chosen in \mathcal{H} and therefore \mathcal{H}/H is connected.

We usually write $H = H^u L$ for a Levi decomposition. When $H = H^u L \subset \mathcal{H} = \mathcal{H}^u \mathcal{L}$ we usually assume $L \subset \mathcal{L}$.

The wonderful G -morphism ϕ is minimal if it does not properly factorize into the composition of two wonderful G -morphisms.

Suppose ϕ to be minimal, and choose $H \subset \mathcal{H}$ as above: we call this a minimal co-connected inclusion of wonderful subgroups.

We have three types of minimal co-connected inclusions ([BL09]):

- (\mathcal{P}) $H^u \supsetneq \mathcal{H}^u$, H is a maximal parabolic subgroup of \mathcal{H} ,
- (\mathcal{R}) $H^u = \mathcal{H}^u$, H is maximal very reductive in \mathcal{H} (i.e. it is contained in no proper parabolic subgroup)
- (\mathcal{L}) $H^u \subsetneq \mathcal{H}^u$, $\text{Lie } H^u$ is a co-simple L -submodule of $\text{Lie } \mathcal{H}^u$, $L = N_L(H^u)$ and L, \mathcal{L} differ only by their connected centers.

Example 3.6.

(1)



This quotient corresponds to the inclusion of H in $\mathcal{H} = Q_-$, minimal parabolic subgroup of $\text{PSL}(4)$, with the same semisimple part $\text{SL}(2)$ and $\text{Lie } H^u$ co-simple $\text{SL}(2)$ -submodule of codimension 2 and in general position in $\text{Lie } Q_-^u$.

(2)



This quotient corresponds to the inclusion of the same H as above in $\mathcal{H} = \text{PSp}(4)$: notice that H is a parabolic subgroup of $\text{PSp}(4)$.

(3)



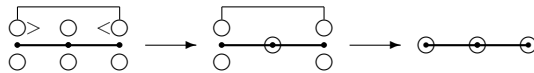
The subgroup $H = \text{PSp}(4)$ is very reductive in $\mathcal{H} = \text{PSL}(4)$.

(4)



Here the quotient corresponds to the inclusion of H as a parabolic subgroup of \mathcal{H} , the minimal parabolic subgroup of $\text{PSp}(4)$ described in (1) and (2). Thus, H is a Borel subgroup of $\text{PSp}(4)$.

(5)



The subgroup H , as in (4), is included in a Borel subgroup B_- of $\mathrm{PSL}(4)$, indeed, these quotients give two minimal co-connected inclusions of type \mathcal{L} : $H \subset \backslash H \subset B_-$.

The type of the minimal co-connected inclusion can be read off the corresponding spherical systems (see loc.cit.). In particular, recall that $d(X) = \mathrm{card} \Delta_X - \mathrm{card} \Sigma_X$ equals the dimension of the connected center of H , therefore it is immediate to see that a minimal co-connected inclusion is of type \mathcal{P} if and only if $d(\backslash X) - d(X) < 0$, actually equal to -1 by minimality. If $d(\backslash X) - d(X) > 0$ then it is of type \mathcal{L} . If it is of type \mathcal{R} then $d(\backslash X) - d(X) = 0$.

The distinguished subset Δ^* is called homogeneous if $\Sigma = \emptyset$ (in this case the corresponding $\backslash X$ is homogeneous).

A minimal homogeneous subset $\Delta^* \subset \Delta_X$ thus gives a parabolic subgroup $Q_- \subset G$ (containing B_-) such that a generic isotropy group H of X can be chosen in Q_- , therefore no other parabolic subgroup of G properly included in Q_- can contain H . When we have such an inclusion $H \subset Q_-$ we say that Q_- is adapted for H . One has $H^u \subset Q_-^u$.

The structure of a general wonderful subgroup of G is essentially described by the following.

Theorem 3.7 ([Lo09], Lemma 4.3.4(2)). *Let H be a wonderful subgroup of G , and choose $Q_- = Q_-^u L_Q$ adapted for H . Then there exists a (unique up to L_Q -conjugation) wonderful subgroup H_* , very reductive in Q_- , such that the inclusion $H \subset H_*$ is of type \mathcal{L} (i.e. composition of minimal co-connected inclusions of type \mathcal{L}).*

3.4. Spherically closed subgroups. Let \tilde{G} be connected and reductive.

Let H be a spherical subgroup of \tilde{G} and $\Delta_{G/H}$ its set of colors. The normalizer $N(H)$ acts on $\Delta_{G/H}$, indeed, $BH = BN(H)$.

Definition 3.8. The stabilizer \overline{H} of the action of $N(H)$ on $\Delta_{G/H}$ is called the spherical closure of H in G . If $H = \overline{H}$, we say that H is spherically closed.

It follows that:

- The spherical closure of a spherical subgroup is spherically closed.
- A self-normalizing spherical subgroup is clearly spherically closed.
- A spherically closed subgroup \overline{H} of \tilde{G} always contains the center of \tilde{G} , therefore the space \tilde{G}/\overline{H} can be thought as a homogeneous space under the action of an adjoint group (\tilde{G} modulo its center).
- The set of colors of G/\overline{H} can be identified with the set of colors of G/H .

The following result is deeper, it may be seen as a partial converse of Theorem 1.4 and is crucial in relating the classification of wonderful varieties and the classification of general spherical varieties (see Appendix B).

Theorem 3.9 ([Kn], Theorem 1.2). *A spherically closed subgroup is wonderful.*

Let $\mathcal{S} = (S^p, \Sigma, A)$ be a spherical G -system. The subset of loose spherical roots Σ_ℓ consists of spherical roots that can be doubled, compatibly with the Cartan pairing. More explicitly, a spherical root σ is loose if:

- $\sigma \in S$ and $c(D_\sigma^+, \sigma') = c(D_\sigma^-, \sigma')$ for all $\sigma' \in \Sigma$, or
- $\mathrm{supp} \sigma$ is of type B_n , $\sigma = \alpha_1 + \dots + \alpha_n$ as in Table 3.1, and $\alpha_n \in S^p$, or
- $\mathrm{supp} \sigma$ is of type G_2 and $\sigma = 2\alpha_1 + \alpha_2$ as in Table 3.1.

Let H be a wonderful subgroup, then the group $N(H)/H$ is direct product of subgroups of order two in bijective correspondence with elements of Σ_ℓ ([Lo09]).

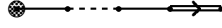
A wonderful subgroup H is spherically closed if Σ_ℓ is included in S .

Example 3.10.

- (1) The maximal torus T of $\mathrm{PSL}(2)$ is spherically closed but not self-normalizing: $\mathrm{PSL}(2)/T$ is of rank 1, with the simple root as spherical root.



- (2) The subgroup $\mathrm{SO}(2n)$ is wonderful but not spherically closed in $\mathrm{SO}(2n + 1)$: $\mathrm{SO}(2n + 1)/\mathrm{SO}(2n)$ is of rank 1, with the highest short root $\alpha_1 + \dots + \alpha_n$ as spherical root.



To conclude it is worth remarking that:

Proposition 3.11 ([BL09], 2.4.2). *Spherically closed subgroups are exactly those arising as isotropy groups of spherical orbits in simple projective spaces.*

The classification of spherically closed subgroups gives the classification of spherical orbits in simple projective spaces, and vice versa (see loc.cit.).

APPENDIX A. SPHERICAL ROOTS

Essential references for this short appendix are [Bri90] and [Kn].

Let G be connected and reductive. Let H be just a spherical subgroup of G and denote by $\mathbb{C}[G]^{(H)}$ the subring of right- H -semiinvariant regular functions on G . As a G -module, $\mathbb{C}[G]^{(H)}$ is multiplicity-free and the occurring highest weights are sometimes called quasi-spherical weights.

Consider

$$m: \mathbb{C}[G]^{(H)} \otimes \mathbb{C}[G]^{(H)} \rightarrow \mathbb{C}[G]^{(H)},$$

the G -equivariant morphism given by the multiplication on $\mathbb{C}[G]$, it is the sum of morphisms $m_\nu^{\lambda, \mu}: V(\lambda) \otimes V(\mu) \rightarrow V(\nu)$, where λ, μ, ν are quasi-spherical weights such that $\lambda + \mu - \nu \in \mathbb{N}S$. Consider the set

$$\mathcal{M}' = \{\lambda + \mu - \nu : m_\nu^{\lambda, \mu} \neq 0\},$$

the saturation of $\mathbb{N}\mathcal{M}'$ in $\mathbb{Z}\mathcal{M}'$ is free, call $\Sigma_{\mathbb{N}}$ its basis.

An element $\eta \in \mathrm{Hom}(\Lambda(G/H), \mathbb{Q})$ lies in the cone of G -invariant valuations $\mathcal{V}(G/H)$ if and only if $\langle \eta, \sigma \rangle \leq 0$ for all $\sigma \in \mathcal{M}'$. Therefore, the elements of $\Sigma_{\mathbb{N}}$ are just equations of the (hyper-)faces of $\mathcal{V}(G/H)$.

Let Σ be the set of primitive equations of $\mathcal{V}(G/H)$ in $\Lambda(G/H)$. The elements of $\Sigma_{\mathbb{N}}$ can be the same or the double of elements of Σ and one exactly has $\Sigma \setminus \Sigma_{\mathbb{N}} = \Sigma_{\ell}$, the set of loose spherical roots (as defined in 3.4).

The quotient $\mathrm{N}(H)/H$ is isomorphic to $\mathrm{Hom}(\Lambda(G/H)/\mathbb{Z}\Sigma_{\mathbb{N}}, \mathbb{C}^\times)$. If $\mathrm{N}(H)/H$ is finite (equivalently $\mathcal{V}(G/H)$ is strictly convex, hence simplicial), denote with X the canonical embedding of G/H . The set Σ corresponds to the set of T -weights in $T_z X/T_z G.z$, where $z \in X$ is the unique point stabilized by B_- (as defined in 1.2).

One has that X is smooth (hence wonderful) if and only if Σ generates $\Lambda(G/H)$. In particular, if $H = \mathrm{N}(H)$, $\Sigma_{\mathbb{N}} = \Sigma$ and X is wonderful.

In general $\mathcal{V}(X)$ is the fundamental chamber of a Weyl group $W(X)$, the so-called little Weyl group of X , and both Σ and $\Sigma_{\mathbb{N}}$ are basis of a root system.

APPENDIX B. SPHERICAL VARIETIES

As already explained in [P], the Luna-Vust theory ([LV]) classifies the equivariant embeddings of any given spherical homogeneous space. By a result in [Lu01] the classification of spherical homogeneous spaces can be reduced to the classification of spherical homogeneous spaces G/H with H spherically closed (recall that a spherically closed subgroup is wonderful, as stated in Theorem 3.9).

Here we give the statement, after a slight reformulation, of Luna's result in loc.cit.

Let H be a spherically closed subgroup of G , X the wonderful embedding of G/H , $\mathcal{S}_X = (S^p, \Sigma, A)$ its spherical system, Δ its set of colors. Define Σ_t to be the set of spherical roots that can be divided by 2 but are not the double of a simple root. Then, spherical subgroups of G with spherical closure equal to H are in correspondence with lattices $\Lambda' \subset \Lambda$ endowed with a pairing $c': \mathbb{Z}\Delta \times \Lambda' \rightarrow \mathbb{Z}$ such that $\Lambda' \supset \Sigma$, $\Sigma \setminus \Sigma_t$ are primitive elements of Λ' , c' extends c and

$$(A2) \text{ for all } \alpha \in S \cap \Sigma, c'(D_\alpha^+, \xi) + c'(D_\alpha^-, \xi) = \langle \alpha^\vee, \xi \rangle \text{ for all } \xi \in \Lambda',$$

- ($\Sigma 1$) for all $2\alpha \in 2S \cap \Sigma$, $\langle \alpha^\vee, \xi \rangle \in 2\mathbb{Z}$ for all $\xi \in \Lambda'$,
- ($\Sigma 2$) for all orthogonal $\alpha, \beta \in S$ with $\alpha + \beta \in \Sigma$, $\langle \alpha^\vee, \xi \rangle = \langle \beta^\vee, \xi \rangle$ for all $\xi \in \Lambda'$,
- (S) for all $\alpha \in S^p$, $\langle \alpha^\vee, \xi \rangle = 0$ for all $\xi \in \Lambda'$.

The above correspondence is realized by associating with the spherical subgroup H' the natural pairing $c_{G/H'} : \mathbb{Z}\Delta_{G/H'} \times \Lambda(G/H') \rightarrow \mathbb{Z}$.

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