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Ivan Losev

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Uniqueness properties for spherical varieties

Ivan Losev

Abstract

The goal of these lectures is to explain speaker's results on uniqueness properties of spherical varieties. By a uniqueness property we mean the following. Consider some special class of spherical varieties. Define some combinatorial invariants for spherical varieties from this class. The problem is to determine whether this set of invariants specifies a spherical variety in this class uniquely (up to an isomorphism). We are interested in three classes: smooth affine varieties, general affine varieties, and homogeneous spaces.

Contents

1. Main results

First of all, let us fix some notation. Throughout the notes, the base field $\mathbb{K}$ is algebraically closed and of characteristic 0. Let G denote a connected reductive algebraic group, B a Borel subgroup in G, and T a maximal torus in B. Then the character lattices $\mathfrak{X}(T), \mathfrak{X}(B)$ of T and B, respectively, are canonically identified. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ be the corresponding Lie algebras.

General goal: given some class C of spherical G-varieties, establish some combinatorial invariants of a variety in C such that this variety is uniquely determined by its combinatorial invariants.

Perhaps, a combinatorial invariant, which is the easiest to define, is the *weight monoid.* Let X be a spherical G-variety. Then $\mathbb{K}[X]$ is a multiplicity free G-module, that is, the multiplicity of every irreducible module in $\mathbb{K}[X]$ is at most 1. By the *weight monoid* $\mathfrak{X}^+_{G,X}$ of X we mean the set of all highest weights of the G-module $\mathbb{K}[X]$.

Exercise 1. Using the fact that $\mathbb{K}[X]$ is an integral domain check that $\mathfrak{X}^+_{G,X}$ is indeed a submonoid in $\mathfrak{X}(T)$.

The following result was conjectured by Knop in the middle of 90's.

Theorem 1 ([Lo1])**.** *Let X_1, X_2 be* smooth *affine spherical varieties with $\mathfrak{X}^+_{G,X_1} = \mathfrak{X}^+_{G,X_2}$. Then X_1, X_2 are G-equivariantly isomorphic.*

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Note that if $G = T$, then the claim of Theorem 2 holds even without the smoothness assumption. Indeed, in this case $\mathbb{K}[X_i] \cong \mathbb{K}[\mathfrak{X}^+_{G,X_i}]$.

However, for $G \neq T$ the theorem no longer holds if we omit the smoothness condition. Indeed, consider the tautological $G := \mathrm{SO}(3)$-module $\mathbb{K}^3$. Let q be a G-invariant non-degenerate quadratic form on $\mathbb{K}^3$. Consider q as a map $\mathbb{K}^3 \to \mathbb{K}$ and let X_0, X_1 be the fibers of 0 and 1, respectively.

Exercise 2. Show that $\mathbb{K}[X_0] \cong \mathbb{K}[X_1]$ as G-modules and, more precisely, that any G-module occurs both in $\mathbb{K}[X_0], \mathbb{K}[X_1]$ with multiplicity 1.

But, of course, X_0, X_1 are not isomorphic as algebraic varieties, for X_1 is smooth, but X_0 is not.

To remedy the situation one needs to consider a more subtle invariant of spherical varieties: the valuation cone, see [Kn1], Corollary 1.8 and Lemma 5.1, or [T], Section 15 and 21. We denote the valuation cone of a spherical G-variety X by $\mathcal{V}_{G,X}$.

Theorem 2 ([Lo1])**.** *Let X_1, X_2 be two affine spherical G-varieties such that $\mathfrak{X}^+_{G,X_1} = \mathfrak{X}^+_{G,X_2}$ and $\mathcal{V}_{G,X_1} = \mathcal{V}_{G,X_2}$. Then X_1, X_2 are G-equivariantly isomorphic.*

Finally, and, in a sense, most importantly, there is a uniqueness property for spherical homogeneous spaces. Here we need three invariants of a spherical G-variety X. The simplest one is the *weight lattice* $\mathfrak{X}_{G,X}$ consisting of all weights of B in the field $\mathbb{K}(X)$ of rational functions, see [Kn1], the paragraph after the proof of Theorem 1.7, or [T], Section 15. The second invariant is the set of B-stable prime divisors of X denoted by $\mathcal{D}_{G,X}$. This is a finite set and we equip it with two maps. The first one maps $\mathcal{D}_{G,X}$ to $\mathfrak{X}^*_{G,X} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}_{G,X}, \mathbb{Z})$, see [Kn1], Section 2, page 8, or [T], Section 15 (in both these papers the map is denoted by ρ). We denote the image of D by φ_D. The second map we need maps $D \in \mathcal{D}_{G,X}$ to its (set-wise) stabilizer $G_D \subset G$. By definition, G_D is a parabolic subgroup in G containing B. Finally, the last invariant we need is $\mathcal{V}_{G,X}$.

The following theorem was, essentially, conjectured by Luna, [Lu1].

Theorem 3 ([Lo2])**.** *Let X_1, X_2 be spherical homogeneous spaces such that $\mathfrak{X}_{G,X_1} = \mathfrak{X}_{G,X_2}, \mathcal{D}_{G,X_1} = \mathcal{D}_{G,X_2}, \mathcal{V}_{G,X_1} = \mathcal{V}_{G,X_2}$. Then X_1, X_2 are G-equivariantly isomorphic.*

The equality $\mathcal{D}_{G,X_1} = \mathcal{D}_{G,X_2}$ requires some explanation (the other two are just equalities of subsets in some ambient set). This equality means that there is a bijection $\iota : \mathcal{D}_{G,X_1} \to \mathcal{D}_{G,X_2}$ with $\varphi_{\iota(D)} = \varphi_D$ and $G_{\iota(D)} = G_D$. However, let us note that even if such a bijection exists it is, in general, not unique. Indeed, any G-equivariant automorphism φ of X_2 induces the bijection $\varphi_* : \mathcal{D}_{G,X_2} \to \mathcal{D}_{G,X_2}$ intertwining the two maps. So we can compose ι with φ_*. It turns out (this is a non-trivial result) that any two bijections ι differ by some φ_*.

Remark 1. It is often more convenient to deal with the system of spherical roots of X. It can be defined as follows. It is known from the work of Brion, [Br2], see also [T], Section 22, that $\mathcal{V}_{G,X}$ is a Weyl chamber for the action of a finite reflection group $W_{G,X}$ (the Weyl group of X) on $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}_{G,X}, \mathbb{Q})$. So we can take linearly independent primitive elements $\alpha_1, \ldots, \alpha_k \in \mathfrak{X}_{G,X}$ such that $\mathcal{V}_{G,X}$ is given by the inequalities $\alpha_i \leqslant 0$. The set $\{\alpha_1, \ldots, \alpha_k\}$ is called the system of spherical roots of X, we will denote it by $\Psi_{G,X}$. If $\mathfrak{X}_{G,X}$ is specified, then $\mathcal{V}_{G,X}$ can be recovered from $\Psi_{G,X}$ and vice versa.

In what follows I will call $\mathfrak{X}_{G,X}, \mathcal{D}_{G,X}, \Psi_{G,X}$ the basic combinatorial invariants of the spherical G-variety X.

To finish the section let us consider an application of Theorem 1, which, in fact, motivated Knop to make his conjecture. This application is the Delzant conjecture from the theory of Hamiltonian actions of compact groups, a reader is referred to [GS] for definitions.

Let K be a connected compact Lie group and $\mathfrak{k}$ be the Lie algebra of K. Fix a maximal torus $T_K \subset K$ and let $\mathfrak{t}_K$ denote the corresponding Lie algebra. Fix a Weyl chamber $C \subset \mathfrak{t}_K$. Our goal is, again, to present combinatorial invariants separating *multiplicity free* Hamiltonian K-manifolds. Recall that one of the equivalent definitions of a multiplicity free compact Hamiltonian K-manifold M is that a general orbit of K on M is a coisotropic submanifold.

Let M denote a multiplicity free compact Hamiltonian manifold with moment map $\mu : M \to \mathfrak{k}$. Recall the moment polytope $\Delta(M) = \mu(M) \cap C$. This is the first invariant we need. The second one is the so called *principal isotropy group*, which will be denoted by $K(M)$. It is defined as the stabilizer of a general point $x \in \mu^{-1}(C)$. It turns out that this stabilizer does not depend on the choice of x.

Conjecture 1 (Delzant)**.** *Let M_1, M_2 be multiplicity free compact Hamiltonian K-manifolds such that $\Delta(M_1) = \Delta(M_2)$ and $K(M_1) = K(M_2)$. Then M_1, M_2 are K-equivariantly symplectomorphic.*

This conjecture was proved by Delzant himself, [D], in the case where K has rank 2. In the general case Knop derived this conjecture from his own in mid 90's, however the proof was never published. In a sentence, the relation between the two conjectures is that Knop's is a local version of Delzant's.

2. Sketch of reduction of the affine case to the homogeneous case

The proof of all three main theorems is based on inductive arguments. We basically have two kinds of inductive steps. One works for homogeneous spaces only and is based on Knop's theory of inclusions of spherical subgroups, [Kn1], Section 4, the other works for all varieties and is based on the Brion-Luna-Vust local structure theorem. We will explain a variant of this theorem due to Knop, [Kn3].

Theorem 4. *Let X be some normal G-variety and $\widetilde{D}$ be a B-stable effective Cartier divisor. Let P be the stabilizer of $\widetilde{D}$ and X^0 denote the complement of $\widetilde{D}$ in X. Finally, let M be the Levi subgroup of P containing T so there is the Levi decomposition $P = M \leftthreetimes \mathrm{R}_u(P)$. Then there is an M-stable subvariety $\Sigma \subset X^0$ such that the natural morphism $\mathrm{R}_u(P) \times \Sigma \to X^0, (p, s) \mapsto ps$, is an isomorphism.*

It is easy to see that Σ is M-spherical provided X is G-spherical. Also one can check that Σ is affine provided X is. The latter follows from the general fact that a complement to a divisor in an affine variety is affine provided the divisor is Cartier.

Actually, one can recover combinatorial invariants of Σ from those of X:

(A) $\mathfrak{X}_{M,\Sigma} = \mathfrak{X}_{G,X}$.
(B) As an abstract set, $\mathcal{D}_{M,\Sigma} = \mathcal{D}_{G,X} \setminus \mathcal{D}$, where $\mathcal{D}$ is the set of irreducible components of $\widetilde{D}$. For $D \in \mathcal{D}_{M,\Sigma}$ we have $M_D = M \cap G_D$ and the vector φ_D is the same as before.
(C) $\Psi_{M,\Sigma}$ is the intersection of $\Psi_{G,X}$ with the linear span of the root system $\Delta(\mathfrak{m})$.
(D) Suppose X is affine and D is the zero divisor of some B-semiinvariant function $f_\mu \in \mathbb{K}[X]$. Then $\mathfrak{X}^+_{M,\Sigma} = \mathfrak{X}^+_{G,X} + \mathbb{Z}\mu$.

Exercise 3. Prove (A),(B),(D).

Our strategy in the proofs of Theorems 1,2 is to reduce them to Theorem 3. We will concentrate on Theorem 1 from now on. We need to check the following two claims:

(*) Let X_1, X_2 be smooth affine spherical varieties such that $\mathfrak{X}^+_{G,X_1} = \mathfrak{X}^+_{G,X_2}$. Then $\mathcal{D}_{G,X_1} = \mathcal{D}_{G,X_2}$.
(**) Let X_1, X_2 be as in (*), so $\mathcal{D}_{G,X_1} = \mathcal{D}_{G,X_2}$. Then $\Psi_{G,X_1} = \Psi_{G,X_2}$.

Once (*) and (**) are proved we can deduce Theorem 1 from Theorem 3 as follows.

Proof of Theorem 1. Let X_1^0, X_2^0 be the open G-orbits in X_1, X_2. One can easily recover the basic combinatorial invariants of X_i^0 from those of X_i.

Exercise 4. Do it.

It follows that X_1^0, X_2^0 satisfy the conditions of Theorem 3 and so are isomorphic. Identify $\mathbb{K}[X_1^0] \cong \mathbb{K}[X_2^0]$. Then $\mathbb{K}[X_1] = \mathbb{K}[X_2]$ since both are the sums of all $V(\lambda) \subset \mathbb{K}[X_i^0]$ with $\lambda \in \mathfrak{X}^+_{G,X_i}$. □

Now we will sketch the proof of (*). An essential ingredient of the proof is the property (D) above.

Namely, choose noninvertible $\mu \in \mathfrak{X}^+_{G,X_i}$. Let $X_1(\mu), X_2(\mu)$ be the M-varieties obtained from X_1, X_2 by using the local structure theorem (here M is the stabilizer of μ in G). Then there exists a bijection $\iota_\mu : \mathcal{D}_{M,X_1(\mu)} \to \mathcal{D}_{M,X_2(\mu)}$ with the required properties. As we mentioned above, $\mathcal{D}_{M,X_i(\mu)}$ is a subset of $\mathcal{D}_{G,X_i}$. It consists precisely of those $D \in \mathcal{D}_{G,X_i}$ s.t. $\langle \varphi_D, \mu \rangle = 0$. In a sense, using the local structure theorem, we can "reveal" all divisors $\mathcal{D} \in \mathcal{D}_{G,X_i}$ such that $\langle \varphi_D, \mu \rangle = 0$ for some noninvertible $\mu \in \mathfrak{X}^+_{G,X_i}$. This motivates the following definition.

Definition 2. An element $D \in \mathcal{D}_{G,X_i}$ is called *hidden* if $\langle \varphi_D, \mu \rangle > 0$ for all $\mu \in \mathfrak{X}^+_{G,X_i} \setminus -\mathfrak{X}^+_{G,X_i}$.

As the following example shows, hidden divisor do occur.

Example 3. Let $G = \mathrm{SL}(n), H = \mathrm{GL}(n-1) \subset \mathrm{SL}(n)$. Then $\mathcal{D}_{G,G/H}$ consists of two elements, and $\mathfrak{X}_{G,G/H}$ has rank 1. Both elements of $\mathcal{D}_{G,G/H}$ are hidden.

Exercise 5. Use an embedding $G/H \hookrightarrow \mathbb{P}(\mathbb{K}^n) \times \mathbb{P}(\mathbb{K}^{n*})$ to prove all claims of the previous example.

There are also other examples, but this one is the most nontrivial, and, in a sense, everything else reduces to it.

So to construct a bijection $\iota : \mathcal{D}_{G,X_1} \to \mathcal{D}_{G,X_2}$ we need:

- To compose different bijections ι_μ together (they do not necessarily agree on intersections). This is relatively easy.
- To show that one actually has coincidence of stabilizers in G not in different M's.
- To get reasonable description of all cases with hidden divisors and deal with them.

The last two parts are quite difficult. We are not going to provide details here.

3. Sketch of the proof in the homogeneous case

In this section we will provide a sketch of the proof of Theorem 2.

As I mentioned before the proof is based on two types of induction.

1. Local structure theorem. Let me explain how to apply Theorem 4 in this case.

Let $H \subset G$ be a spherical subgroup that can be included into a proper parabolic subgroup. Conjugating, we may assume that H is contained in a parabolic subgroup Q^- that is opposite to B. Let M be the Levi subgroup of Q^- containing T and $Q := BM$. Consider the projection $\pi : G/H \twoheadrightarrow G/Q^-$. Let $\widetilde{D}_0$ be the complement to the open Q-orbit in G/Q^-. Then $\widetilde{D}_0$ is a divisor, so we can take $\widetilde{D} = \pi^{-1}(\widetilde{D}_0)$ in Theorem 4.

Exercise 6. Show that one can take Q^-/H for the section Σ.

In general, the spherical variety Q^-/H is hard to deal with. However, there are cases when this variety is affine. Namely, let $H = S \leftthreetimes N$ be a Levi decomposition. Suppose that N is contained in the unipotent radical $\mathrm{R}_u(Q^-)$ of Q^- (this is always the case when Q^- is a minimal parabolic containing H). Then we can conjugate H by an element from $\mathrm{R}_u(Q^-)$ and assume that $S \subset M$. In this case Q^-/H is the homogeneous vector bundle $M *_S (\mathrm{R}_u(\mathfrak{q}^-)/\mathfrak{n})$ (over M/S with fiber $\mathrm{R}_u(\mathfrak{q}^-)/\mathfrak{n}$).

2. Knop's theory of inclusions of spherical subgroups. Let $X := G/H$ be a homogeneous space. Then

(1) The set of all subgroups $\widetilde{H} \subset G$ such that $\widetilde{H} \supset H$ and $\widetilde{H}/H$ is connected can be described entirely in terms of $\mathfrak{X}_{G,X}, \mathcal{D}_{G,X}, \mathcal{V}_{G,X}$. Namely, subgroups $\widetilde{H}$ are in one-to-one correspondence with pairs $(\mathfrak{a}, \mathcal{D})$ (so called, *colored subspaces*), where $\mathfrak{a}$ is a subspace in $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{X}_{G,X}, \mathbb{Q})$, $\mathcal{D}$ is a subset in $\mathcal{D}_{G,X}$, satisfying some combinatorial conditions.

(2) Let $\widetilde{H}$ be a subgroup corresponding to a colored subspace $(\mathfrak{a}, \mathcal{D})$. Then the combinatorial invariants of $G/\widetilde{H}$ can be recovered from those for G/H and from $(\mathfrak{a}, \mathcal{D})$.

This claim allows to do induction. Namely let us take two homogeneous spaces $X_1 = G/H_1, X_2 = G/H_2$ satisfying the conditions of Theorem 1. Take a minimal subgroup $\widetilde{H}_1$ containing H_1 properly. This subgroup gives rise to a colored subspace (for X_1). Take the same colored subspace for X_2 and let $\widetilde{H}_2$ be the corresponding subgroup of G containing H_2. But now $G/\widetilde{H}_1$ and $G/\widetilde{H}_2$ have the same basic combinatorial invariants. By inductive assumptions, $\widetilde{H}_1$ and $\widetilde{H}_2$ are conjugate. So we may assume that

(1) $\widetilde{H} = \widetilde{H}_1 = \widetilde{H}_2$,

(2) and the colored subspaces of the inclusions $H_1 \subset \widetilde{H}, H_2 \subset \widetilde{H}$ are the same.

There is a subtlety in the second part coming from the fact that a bijection between the sets of divisors is non-unique, but this can be fixed.

We described the induction steps but did not mention the base. Well, the base is the case when neither H_1 nor H_2 can be included into a proper parabolic. But then H_1, H_2 are both reductive and one can use the classification due to Krämer, Brion and Mikityuk, [Kr], [Br1], [M], to prove Theorem 3.

Now we are ready to give a sketch of the proof of Theorem 3.

Step 1. We may assume that $N_G(H_i)^\circ \subset H_i$ for $i = 1, 2$. This can be deduced from Luna's results, [Lu2].

Step 2. Let $H_1 = S \leftthreetimes N_1$ be a Levi decomposition. Then there exists a subgroup $\widetilde{H}$ containing H_1 and such that $\widetilde{H} = S \leftthreetimes \widetilde{N}$, where $\widetilde{\mathfrak{n}}/\mathfrak{n}_1$ is an irreducible S_1-module. As explained above, we may assume that H_2 is contained in $\widetilde{H}$. Thus $\dim H_2/\operatorname{R}_u(H_2) \leqslant \dim \widetilde{H}/\operatorname{R}_u(\widetilde{H}) = \dim S$. Now from the symmetry between H_2 and H_1 we see that the previous inequality turns into equality so we can write $H_2 = S \leftthreetimes N_2$. Since $\widetilde{H}$ is a minimal subgroup containing H_2 we get that $\widetilde{\mathfrak{n}}/\mathfrak{n}_2$ is an irreducible S-module.

Step 3. Pick a minimal parabolic subgroup Q^- containing $N_G(\widetilde{H})$. Note that $N_1, N_2 \subset \widetilde{N} \subset \operatorname{R}_u(Q^-)$. We may assume that Q^- is opposite to B and that $S \subset M$. From the local structure theorem we deduce that the affine spherical M-varieties $Q/N_1 = M *_S (\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_1), Q/N_2 = M *_S (\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_2)$ have the same basic combinatorial invariants. Therefore, using the induction assumption, we obtain that there is an M-equivariant isomorphism $M *_S (\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_1) \to M *_S (\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_2)$. It follows that the S-modules $\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_1, \operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_2$ are conjugate under the action of $N_M(S)$.

Step 4. Suppose for a moment that $\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_1$ and $\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_2$ are actually isomorphic as S-modules. Then $\widetilde{\mathfrak{n}}/\mathfrak{n}_1, \widetilde{\mathfrak{n}}/\mathfrak{n}_2$ are isomorphic as S-modules. The following exercise implies that $[\widetilde{\mathfrak{n}}, \widetilde{\mathfrak{n}}] \subset \mathfrak{n}_1 \cap \mathfrak{n}_2$.

Exercise 7. Let $\widetilde{\mathfrak{n}}$ be a nilpotent Lie algebra, S an algebraic group acting on $\widetilde{\mathfrak{n}}$ by Lie algebra automorphisms. Let $\mathfrak{n}$ be an S-stable subalgebra in $\widetilde{\mathfrak{n}}$ such that $\widetilde{\mathfrak{n}}/\mathfrak{n}$ is an irreducible S-module. Then $[\widetilde{\mathfrak{n}}/\widetilde{\mathfrak{n}}] \subset \mathfrak{n}$.

Now since $[\widetilde{\mathfrak{n}}, \widetilde{\mathfrak{n}}] \subset \mathfrak{n}_1 \cap \mathfrak{n}_2$ and $\widetilde{\mathfrak{n}}/\mathfrak{n}_1, \widetilde{\mathfrak{n}}/\mathfrak{n}_2$ are S-equivariantly isomorphic, there is a family $\mathfrak{n}(t), t \in \mathbb{P}^1$, of S-stable subspaces of $\widetilde{\mathfrak{n}}$ containing $\mathfrak{n}_1 \cap \mathfrak{n}_2$ and such that $\mathfrak{n}(0) = \mathfrak{n}_1, \mathfrak{n}(\infty) = \mathfrak{n}_2$. So $\mathfrak{h}_1$ can be deformed to $\mathfrak{h}_2$. Now the rigidity results of Alexeev and Brion, [AB], together with the assumption of step 1 imply that $\mathfrak{h}_1, \mathfrak{h}_2$ are G-conjugate. Using the equality $\mathfrak{X}_{G,X_1} = \mathfrak{X}_{G,X_2}$ one can check that H_1, H_2 are also conjugate.

Step 5. Recall that we still have not proved that the S-modules $\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_1, \operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_2$ are actually isomorphic. We only checked that they are conjugate under the action of $N_M(S)$ on the set of S-modules. So pick an element $\gamma \in N_M(S)$ that conjugates $\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_2$ to $\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_1$. One can show that γ^2 acts trivially on the set of modules. It follows that γ fixes $\operatorname{R}_u(\mathfrak{q}^-)/\widetilde{\mathfrak{n}}$. So γ can be lifted to an automorphism of $M *_S (\operatorname{R}_u(\mathfrak{q}^-)/\widetilde{\mathfrak{n}})$. Now the crucial observation is that γ can be actually lifted to an element $g \in N_G(\widetilde{H})$. This follows from the description of the group of equivariant automorphisms of a spherical variety. This description will be discussed briefly in the next section. Replacing H_2 with gH_2g^{-1} we obtain $\operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_1 \cong \operatorname{R}_u(\mathfrak{q}^-)/\mathfrak{n}_2$.

This completes the proof.

4. Equivariant automorphisms and Demazure embeddings

We have already mentioned that the group of equivariant automorphisms $\operatorname{Aut}^G(X)$ of a spherical G-variety X can be recovered (at least, when X is affine or homogeneous) from the basic combinatorial invariants. Let us state this description here.

At the first glance, the group $\operatorname{Aut}^G(X)$ has nothing to do with $\mathfrak{X}_{G,X}, \mathcal{D}_{G,X}, \Psi_{G,X}$. However, Knop, [Kn4], discovered a relation. Let us recall that to any $\lambda \in \mathfrak{X}_{G,X}$ we have assigned a unique (up to rescaling) rational function $f_\lambda \in \mathbb{K}(X)$. Pick $\varphi : \operatorname{Aut}^G(X)$. Then $\varphi.f_\lambda$ is again B-semiinvariant of weight λ. So there is a unique scalar $a_{\varphi,\lambda}$ such that $\varphi.f_\lambda = a_{\varphi,\lambda} f_\lambda$.

Exercise 8. The map $a_\varphi : \mathfrak{X}_{G,X} \to \mathbb{K}^\times, a_\varphi(\lambda) = a_{\varphi,\lambda}$ is a character of $\mathfrak{X}_{G,X}$.

Define the *root lattice* $\Lambda_{G,X} = \bigcap_{\varphi\in \mathrm{Aut}^G(X)} \ker a_\varphi$. The following exercise explains the terminology.

Exercise 9. Let $X := H$ be a connected reductive algebraic group, and let $G := H \times H$ act on H by two-sided multiplications. Then $\mathfrak{X}_{G,X}$ is identified with the weight lattice of H. Show that $\Lambda_{G,X}$ is the root lattice of H.

Similarly, one can define the homomorphisms $a_\lambda : \mathrm{Aut}^G(X) \to \mathbb{K}^\times$ for $\lambda \in \mathfrak{X}_{G,X}$. As Knop proved in [Kn4], $\bigcap_{\lambda\in\mathfrak{X}_{G,X}} \ker a_\lambda = \{1\}$. So the group $\mathrm{Aut}^G(X)$ is identified with $(\mathfrak{X}_{G,X}/\Lambda_{G,X})^*$. Therefore to describe $\mathrm{Aut}^G(X)$ it will be enough to describe the subgroup $\Lambda_{G,X} \subset \mathfrak{X}_{G,X}$.

We will do slightly better: we will describe a distinguished basis in $\Lambda_{G,X}$. Namely, one can construct a set of vectors $\overline{\Psi}_{G,X} \subset \Lambda_{G,X}$ completely analogously to the construction $\Psi_{G,X} \subset \mathfrak{X}_{G,X}$. Knop proved in [Kn4] that $\overline{\Psi}_{G,X}$ is a basis in $\Lambda_{G,X}$.

Clearly, any element in $\overline{\Psi}_{G,X}$ is proportional to exactly one element on $\Psi_{G,X}$. In fact, all coefficients are either 1 or 2. Theorem 2 from [Lo2] explicitly explains how we distinguish between 1 and 2 analyzing $\Psi_{G,X}$ and $\mathcal{D}_{G,X}$. The description is technical and therefore we omit it (however, see Conjecture 4 in the next section; in the spherical case it boils down to the description).

Instead we will explain a result, which motivated us to state and prove Theorem 2 from [Lo2] (applications to the proof of Theorem 3 were discovered later). This result is the conjecture of Brion, [Br2], that Demazure embeddings are smooth.

Let us explain what "Demazure embedding" means. Take a self-normalizing spherical subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then we can consider the G-orbit $G\mathfrak{h}$, where $\mathfrak{h}$ is viewed as a point in the Grassmann variety $\mathrm{Gr}(\mathfrak{g})$. Take the closure $\overline{G\mathfrak{h}}$. Brion conjectured that the closure is smooth.

The first indication that the Brion conjecture can be related to the description of $\mathrm{Aut}^G(X)$ is as follows. Let us note that the open G-orbit in $\overline{G\mathfrak{h}}$ is nothing else but $G/N_G(\mathfrak{h})$. It is easy to notice that this homogeneous space has no nontrivial G-equivariant automorphisms.

Exercise 10. Show it.

In [Kn4] Knop proved that the *normalization* of the Demazure embedding is smooth. Brion's conjecture was proved by Luna for type A in [Lu3]. Using the description of $\mathrm{Aut}^G(X)$ the author was able to extend Luna techniques to the general case, see [Lo3].

5. Generalizations

All three theorems, as well as the description of the group of equivariant automorphisms, have interesting conjectural generalizations to the general (not necessarily spherical) case. The invariants appearing in these generalizations are mostly quite difficult to define and are even more difficult to deal with.

Again, the conjecture that is easiest to state deals with smooth affine varieties. For a generalization of a weight monoid $\mathfrak{X}^+_{G,X}$ we take the algebra $\mathbb{K}[X]^U$ of U-invariants equipped with the natural T-action. Here U is the maximal unipotent subgroup of B. The algebras of U-invariants were extensively studied in Invariant theory.

Exercise 11. Check that if X if an affine spherical variety, then $\mathbb{K}[X]^U$ is T-equivariantly isomorphic to the monoid algebra $\mathbb{K}[\mathfrak{X}^+_{G,X}]$.

Conjecture 2. *If X_1, X_2 are smooth affine G-varieties such that the algebras $\mathbb{K}[X_1]^U, \mathbb{K}[X_2]^U$ are T-equivariantly isomorphic, then X_1, X_2 are G-equivariantly isomorphic.*

In general, it is known, see [AB], Corollary 2.9, that there are only finitely many isomorphism classes of affine G-varieties X with a fixed $\mathbb{K}[X]^U$.

Unfortunately, it seems to be unlikely that one will be able to prove this conjecture without proving an analog of Theorem 3. This analog should deal with birational classification of G-varieties instead of just the classification of homogeneous spaces. Therefore let us describe the invariants which (conjecturally) should be used. Below X stands for a normal irreducible G-variety.

For a generalization of the weight lattice $\mathfrak{X}_{G,X}$ we take the set $\mathbb{K}(X)^{(B)}$ of all rational B-semiinvariant functions on X. This set is equipped with the multiplication and partially defined addition, both induced from $\mathbb{K}(X)$.

The other two invariants are now generalized directly. Consider the set $\mathcal{D}_{G,X}$ consisting of all prime Weil divisors on X that are B-stable. Again, there is a natural map $\mathcal{D}_{G,X} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{K}(X)^{(B)\times}/\mathbb{K}^\times, \mathbb{Z})$, $D \mapsto \varphi_D, \varphi_D(f) := \mathrm{ord}_D(f)$. Clearly, this set is not a birational invariant: to get one just consider the set $\mathcal{D}^0_{G,X} = \{D \in \mathcal{D}_{G,X} | G_D \neq G\}$.

Finally, one can consider the set $\overline{\mathcal{V}}_{G,X}$ consisting of all geometric (i.e., coming from divisors, in an appropriate sense) $\mathbb{Q}$-valued discrete G-invariant valuations of $\mathbb{K}(X)$. This set is equipped with the map $\overline{\mathcal{V}}_{G,X} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{K}(X)^{(B)\times}/\mathbb{K}^\times, \mathbb{Q})$ given by $v \mapsto \varphi_v, \varphi_v(f) := v(f)$. This map is known to be injective, see [Kn2], so we consider $\mathcal{V}_{G,X}$ as a subset in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{K}(X)^{(B)\times}/\mathbb{K}^\times, \mathbb{Q})$.

We call the triple $(\mathbb{K}(X)^{(B)\times}, \mathcal{D}^0_{G,X}, \overline{\mathcal{V}}_{G,X})$ the Luna-Vust (shortly, LV) system of X X) and denote it by $\mathcal{LV}_{G,X}$. The reason for this is that these invariants appeared already in [LV]. See [T], Sections 12-14, 20-21, for more information about them.

By an *isomorphism* of two LV systems $(F_1, \mathcal{V}_1, \mathcal{D}_1), (F_2, \mathcal{V}_2, \mathcal{D}_2)$ we mean a pair (ψ, ι), where $\psi : F_1 \to F_2, \iota : \mathcal{D}_1 \to \mathcal{D}_2$ are isomorphisms satisfying the natural compatibility relations. For example, T acts by automorphisms of a LV system $\mathcal{LV} = (F, \mathcal{V}, \mathcal{D})$ (the action on $\mathcal{D}$ is trivial). By $\mathrm{Aut}^{ess}(\mathcal{LV})$ we denote the quotient of the whole group $\mathrm{Aut}(\mathcal{LV})$ by the image of T. In a sense, this group consists of "essential" automorphisms of $\mathcal{LV}$, while automorphisms coming from T are considered as "non-essential".

We would like to make the following two conjectures.

Conjecture 3. *Let X_1, X_2 be two normal irreducible G-varieties with isomorphic LV systems. Then X_1, X_2 are birationally isomorphic (as G-varieties).*

Conjecture 4. *The group of birational G-equivariant automorphisms of X surjects onto* $\mathrm{Aut}^{ess}(\mathcal{LV})$.

Here is a very rough strategy that one can use to prove these conjectures. This strategy is just a direct generalization of the one used in [Lo2]:

Step 1. Reduce Conjectures 3,4 to the case of homogeneous spaces (this step does not occur in the spherical case).

Step 2. Prove Conjectures 3, 4 for affine homogeneous spaces.

Step 3. Develop the theory of inclusions of subgroups of G on the language of LV systems. Such a theory was developed by Knop, [Kn1] in the spherical case.

Step 4. Prove Conjecture 4 generalizing the proof of [Lo2], Theorem 2.

Step 5. Prove Conjecture 3 generalizing the proof of [Lo2], Theorem 1.

Of course, performing (at least some of) these steps is not easy at all. For example, step 2 should involve some new ideas (this step in the spherical case relies heavily on the classification).

After Conjecture 3 is proved one should be able to prove Conjecture 2 by verifying analogs of claims (*),(**) from Section 2.

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MIT, Department of Mathematics, 77 Massachusetts Avenue, Cambridge MA 02139, USA • ivanlosev@math.mit.edu